Research Statement

Timothée Bonnefoi [*][†]

I am currently finishing my PhD at the University of Antwerp under the supervision of Wendy Lowen and Arne Mertens (defence expected in Fall 2026). My research interests are in category theory specifically topos theory, monoidal categories, formal category theory and their links with logic.

I. Formal toposes

A formal theory of toposes My thesis defines formal toposes, that is toposes [AGV71; MM94; Joh02] in the context of formal category theory [Gra74; SW78; Str72; DL23], which allows to do category theory in a general 2-category $\mathcal{B}$. My approach generalises the characterisation of toposes as left-exact localisations of presheaf categories over a small category. This splits the problem into, on the one hand some choice of reference objects (the small presheaf categories) which we may take as given by a Yoneda structure [SW78], and on the other hand a characterisation of subtoposes (the lex localisation).

Pleronomial objects This last problem is solved by observing that the preservation of pullbacks and terminal objects (and hence of all finite limits) can be encoded entirely in terms of the codomain fibration. Therefore my thesis relies on the introduction of a new object that allows to generalise it: the ***pleronomial object***. Given two 1-morphisms $\mathbf{F}, \mathbf{G} : \mathbf{I} \to \mathbf{E}$ of $\mathcal{B}$, it is the universal object $\mathbf{E}^{\mathbf{F}}_{\mathbf{G}}$ equipped with a 1-morphism $\mathbf{H} : \mathbf{E} \to \mathbf{E}^{\mathbf{F}}_{\mathbf{G}}$ and a 2-morphism $\mathbf{HF} \Rightarrow \mathbf{HG}$. This builds on the notion of indeterminate, first appearing in [AGV71] and later formalised in [LS88; HJ95], and generalises cone categories and especially slice categories. From the point of view of logic, this object can be nicely interpreted as extending the logical theory represented by $\mathbf{E}$ with a new (generalised) variable of type/sort $\mathbf{G}$; geometrically it behave like a tangent space of $\mathbf{E}$ at $\mathbf{G}$ but seen by the 1-morphism $\mathbf{F}$.

Toposes Working in a ***topological cosmos***, that is a sufficiently nice 2-category equipped with a Yoneda structure and possessing sufficiently many pleronomials, I have been able to give a definition of toposes. Pleronomial objects, through a generalisation of indexed categories, can be organised into a ***self-indexing*** which extends the codomain indexed category. This allows to define ***toposes*** as reflective subobjects of the reference objects (from the Yoneda structure) whose reflector satisfies coherence conditions with respect to the self-indexing. Those conditions are reminiscent of the Beck-Chevalley condition and notably include that the reflector should induce an indexed 1-morphism.

Current results As a generalisation of toposes, this theory can be judged along two different axes: by its behaviour and by its accordance with the data. The former corresponds to recovering, in the formal setting, properties and characterisations satisfied by Grothendieck toposes. While the latter implies checking whether the theory specialises, in known cases, to the more ad-hoc generalisations appearing in the literature ([Lur08; BQ96] and the many proposals in the monoidal case, see Monoidal toposes).

[*]Currently: Department of Mathematics, University of Antwerp
[†] 0000-0003-1636-0812

Known cases Indeed as this notion of toposes is defined in a general cosmos it can be instantiated in specific cases: in particular it specialises in $\mathcal{C}at$ to the classical notion of Grothendieck toposes and their geometric morphisms [AGV71, Exposé IV] and to locales in $\mathcal{P}os$. But it also seems to recover Lurie's higher toposes [Lur08] in the homotopy 2-category of the $(\infty, 2)$-category of quasicategories and for the slightly different notion of ***accessible*** toposes (preliminary work). This extends [RV22] by showing that toposes too can be described in this 2-category. More examples are being worked on.

Characterisations of toposes Moreover formal toposes satisfy several equivalent characterisations under conditions on the ambient cosmos (conditions all satisfied both in $\mathcal{C}at$ and the other examples). These characterisations include the presentation of toposes as left-exact localisations of presheaf categories over a small category (the characterisation that this definition generalises), as categories of algebras for left-exact idempotent monads on those same categories, and a slight modification of the characterisation by universal closure operators (side-stepping the notion of subobjects). Here too, more are underway.

II. Future Works

The work done so far is only the first step towards a mature formal theory of toposes. On the one hand future works could focus on extending aspects of $\mathcal{C}at$ topos theory in particular the links with logic or geometry, on the other hand they could embrace the new problematics arising from the formal theory either with the study of cosmoi or of possible modifications of the theory. Finally a last line of research is that of monoidal toposes which promise to be one of the most interesting special case in terms of applications.

Formal theory

The classical theory of toposes is rich [Joh02] and the formal theory developed so far only brushes the beginnings of it. To what extent can those results be reproduced ? And what conditions must be put on the cosmos to do so ? Moreover are there other examples of toposes that can be computed ?

Theory The method used, with success, to prove the different characterisations of toposes consists in reformulating the $\mathcal{C}at$-based statements in terms of slice categories and then replace those with the pleronomial objects, possibly adding some assumptions on the ambient cosmos along the way. While this mostly consists in adapting existing proofs, it also allows to reveal the hidden assumptions arising from working in the very nice 2-category $\mathcal{C}at$.

Subobjects A closely related objective is to generalise the notion of subobjects with which a lot of topos theory is expressed. The notion is not well-behaved even in cases close to $\mathcal{C}at$ as evidenced by higher toposes where Grothendieck topologies do not recover all toposes [Ane+22]. A possible method is to reframe the subobjects in terms of the induced base change functors of the self-indexing, while another is to modify the universal property of the pleronomial object into one satisfied by the subobject posets.

Instantiations of the theory There are still many 2-categories of interest where it is not known whether formal toposes exist and, if so, the specific form they take. Important examples are enriched categories (where a theory of sheaves already exists [BQ96]) and monoidal categories (see Monoidal toposes). However experience shows that computing the pleronomial objects can prove particularly difficult. Beyond those specific cases, one may also attempt to prove some general result e.g. for 2-categories of (co)algebras of ((co)KZ)-pseudo(co)monads on a given cosmos.

Logic and type theory

One of the most interesting aspect of toposes is their link with logic and type theory [MM94; LS88] therefore extending this aspect is a natural objective. This avenue of research is very interesting yet open-ended: it is possible that there is nothing, or very little, to say in full generality. However it remains particularly interesting even when restricting the cosmos under consideration to those sufficiently close to $\mathcal{C}at$ (like $(\mathcal{S}ym)\mathcal{M}on\mathcal{C}at$) which should be easier to work with.

Geometric logic From the point of view of logic, Grothendieck toposes are intimately linked with geometric logic [Vic14] whose properties can, essentially, be read in what the inverse image functor of a geometric morphism preserves. The formal theory has its own notion of geometric morphism (which specialises to the classical one in $\mathcal{C}at$), it might therefore be possible to reverse engineer logics which would find categorical semantics (or a generalisation thereof) in formal toposes. If successful, such an approach would also allow to generalise the notion of classifying topos for those theories.

First and higher order logic The other link of toposes with logic is with first and higher order logic [LS88]. However those rely on the notion of subobject and subobject classifier to interpret the lattice of predicates in a given context. Therefore the first step in this direction would be to generalise those. Supposing this done, the same ideas can be repeated here.

Dependent type theory In the same line but with a slightly different flavour, toposes, as locally cartesian closed categories, offer categorical semantics to dependent type theory through the codomain fibration [Jac99]. With the self-indexing, the formal theory offers a generalisation of this fibration in any cosmos (in fact this only relies on the notion of pleronomial objects and not that of toposes). By building upon the notion of comprehension category [Jac93], it could offer a semantic notion of dependent type theory corresponding to the theory of the cosmos, and in particular of linear dependent type theory in $\mathcal{S}ym\mathcal{M}on\mathcal{C}at$. However it is unclear how to transcribe this syntactically in general.

GEOMETRY Toposes have strong links with geometry whether as categories of geometric objects or as spaces themselves (gros/petit topos). These links are as intrinsic as the links with logic and it is therefore important to be able to speak of formal toposes geometrically [Joh02, Part C.]. This is of course related to Geometric logic. This promises new and rather foreign notions of space including non-commutative spaces. In a similar fashion to the logical case the general case might prove difficult but specific examples should be more tractable.

The fundamental theorem and spatial cosmoi One problem of the current formal theory of toposes is that the so-called fundamental theorem of topos theory [McL95] does not hold in general i.e. *a priori* pleronomial objects of toposes are not themselves toposes. This is a significant hurdle for a petit geometric interpretation of formal toposes as, for Grothendieck toposes, étale spaces are defined as the slice toposes and play a significant role in the geometric interpretation [AJ21]. Therefore a first step is to define, with the methods of Theory, *spatial cosmoi* with sufficient conditions to derive the theorem.

Classifying spaces and types of morphisms Beyond the fundamental theorem, a proper geometric interpretation should generalise geometric concepts such as points, opens, quotient, fiber product, etc. Ideally, the theory should have also classifying objects for such notions whenever possible. It is to be expected however that some result or constructions would not generalise cleanly (already the pleronomial has an added parameter compared to the slices) hinting at more exotic notions of geometry.

Sheaves The other point of view concerns sheaf theory which is appears thorough geometry [KS05; Vak17] and beyond. They can be straightforwardly defined in the formal theory, however they remain to be thoroughly studied. In particular it would be interesting to try to understand for what kind of spaces they could be sheaves of functions on. This might be difficult in full generality but special cases may relate to current approaches to non-commutative geometry [Mah06].

Cosmology Cosmoi introduce a new dimension to the theory as the known notions of toposes are all defined within a single cosmos. It is therefore natural to wonder what can be said of cosmoi in general and how changes to them reflect on the toposes therein. This can be split into three questions.

Classes of cosmoi While the known notions of toposes all correspond to cosmoi satisfying strong assumptions where we can recover many results from the $\mathcal{C}at$ case, in the formal theory those assumptions are not always satisfied. This leads to the definition of several different classes of cosmoi. Studying these, how they relate to each other, and what result can be achieved in each case would yield better understanding of formal toposes as a whole.

Cosmoi as 2-categorical objects We can also view cosmoi as 2-categorical objects i.e. in an appropriate tricategory: natural questions are then whether they can be identified by some general axiomatic categorical property and if there are ways to generate new cosmoi from existing ones (e.g. by taking the 2-categories of (co)algebras for a pseudo(co)monad) which would yield more examples of topos theories. One particularly interesting question is the links between cosmoi and 2-toposes in $2-\mathcal{C}at$ (as in defined [Str82] or some other notion), and in particular if cosmoi are always 2-toposes as the microcosm principle [BD98] would suggest.

Change of base Finally, in the same spirit, we can study 'change of base' pseudofunctors between cosmoi and how the notions of pleronomial object and topos behave under such functors, extending change of base for enriched categories [Ros25]. In particular such considerations would allow to better understand, and generalise, the notion of localic topos. This also is interesting as a way to derive constraints on the pleronomial objects when computing them in new settings.

Yet more toposes The formal definition of toposes applies in many different settings and opens a path to recognise many objects as toposes. Yet the theory is not rigid and, keeping the same ideas, it could be modified to adapt itself to different settings. To what extent can the theory be generalised in 2-categories and beyond ? On the contrary can it be strengthened to distinguish particularly well-behaved cosmoi ?

Weakening of the models The current theory does not rely heavily on the axioms of Yoneda structures therefore a natural extension would be to replace Yoneda structures with a weaker notion of reference objects, for example with a (KZ)-pseudomonads (see [Wal18] which shows in particular that faithful KZ-pseudomonads are 'almost' Yoneda structures). While this will undoubtedly weaken the expressive power of the theory, it might also allow to understand other objects as toposes. In particular sheaf subtoposes can still be defined on ***elementary*** toposes [Joh02], so one may wonder if they can be recovered as formal toposes but with respect to a more general notion of reference objects.

Higher toposes Pleronomials can be succinctly defined as the objects freely adding a 2-morphism between the two given 1-morphisms. From this point of view the current theory developed inside 2-categories should, morally, be extendable to the tricategorical context. Albeit instead of defining the pleronomial to freely add a 2-morphism we should, a priori, do so for both 2 and 3-morphisms as subtoposes should be allowed to invert both kinds. It remains unclear what shape this should take practically. Such a notion should be compared to, and informed by, the existing definition of [Str82]. At the limit this could also allow to define (n, r)-toposes for some of the proposed notions of (n, r)-categories.

Formal *elementary* toposes One weakness of the current formulation of pleronomials is that they are only defined on the cocomplete objects with respect to the Yoneda structure. By modifying their definition, taking it closer to its 'category of cones' interpretation, the notion could be weakened to apply to every object, at least in special cases. In such cases moreover one could expect that a natural choice of Yoneda structure would arise from just the 2-categorical structure, hence obtaining a stronger notion of cosmos. Such a reformulation would open the way to define formal ***elementary*** toposes(modulo a notion of subobject) by relying solely on pleronomials (note that this is perpendicular to the approach put forth in Weakening of the models).

Monoidal toposes One particularly interesting special case is that of monoidal categories. Compared to the other research objectives this has a rather different flavour, indeed while it builds on the formal notion of toposes it proposes to focus on one concrete case (as far as category theory goes). This is one of the most exciting avenues of research in terms possible applications to mathematics, physics, complex systems and computer science.

The computation of pleronomials The original motivation for the formal topos theory project was, in fact, the special case where the 2-category of interest was $(\mathcal{S}ym)\mathcal{M}on\mathcal{C}at$ (with some choice of (symmetric) lax/colax/strong monoidal functors, [Mac78]). This problem took a significant part of my PhD and computing the pleronomial object for a general symmetric monoidal category has looked particularly difficult for some time, with results only in special cases so far. But with additional results and understanding from the general formal case, in particular change of cosmoi pseudofunctors, it now seems within grasp.

Comparison to the literature The question of defining sheaves on, and toposes with respect to, a monoidal category has already been explored in the literature. Several sources define sheaves but only when restricting to quantales i.e. (right) closed monoidal locales [BB86; BC93; Hey10; Res12; Hey14] sometimes with further assumptions on the quantale. A more general approach appears in [TAM25; Ten23] where the authors define sheaves on semicartesian (symmetric) monoidal categories. [CHB19] gives an axiomatic definition of non-commutative toposes by demanding the existence of an internal non-commutative frame to replace the subobject classifier (subject to conditions). Finally let us also point out [Amb91] that, while not setting out for sheaves and toposes, has closely related considerations. The notion obtained from the formal theory should be compared with those when it makes sense while attempting to understand why any differences arise.

Theory and logic, again If the pleronomial object is successfully computed then one could take the sections on the formal theory and on logic, and ask the same questions in that case. Here however answers should be much easier to obtain due to the concrete nature of monoidal categories. Notably the links of symmetric monoidal categories with (multiplicative intuitionistic) linear logic [Gir87; Mel09] would guide these developments, in particular the type theoretic questions of the previous section would form a step towards linear dependent type theory (e.g. [Vák15] for a proposition). This case also benefits from the rich literature on monoidal categories and seems to has links with coalgebra theory [Ada05], in particular the appropriate notion of subobjects may be related to coequations [DS21].

Even though such things are still far off, I would be remiss not to mention the particularly interesting applications that are possible in the monoidal case as monoidal categories appear in many places. Among others: in mathematics with various forms of (non-commutative) geometry [Bal10; Ško09; Bra14] and probability theory [Fri20], see also [Str12]; in computer science with the already mentioned dependent linear type theory (of relevance to concurrency), game theory [Gha+18] or deep learning [FST21]; in quantum physics [CK17; HV19; Coe10; SS11] each with a different approach; as well as many examples of applied category theory e.g. to electric circuits [BS22], chemistry [Lob25] or linguistics [Lam12]. It is unlikely that monoidal sheaves or toposes may be of use in all of those areas but they may allow new insights by recognising objects as toposes, sheaves or sites, or new methods by considering sheaves over the monoidal categories appearing in those settings.

References

[Ada05] J Adamek. "Introduction to Coalgebra". In: ***Theory and Applications of Categories*** (2005).

[Amb91] S. Ambler. "First order linear logic in symmetric monoidal closed categories". PhD thesis. University of Edinburg, Department of Computer Science, 1991.

[AJ21] Mathieu Anel and André Joyal. "Topo-logie". In: ed. by Gabriel Catren and Mathieu Anel. Vol. 1. Cambridge: Cambridge University Press, 2021, pp. 155–257. ISBN: 9781108490634. DOI: 10.1017/9781108854429.007.

[Ane+22] Mathieu Anel et al. "Left-exact Localizations of ∞-Topoi I: Higher Sheaves". In: ***Advances in Mathematics*** 400 (May 2022). arXiv:2101.02791 [math], p. 108268. ISSN: 00018708. DOI: 10.1016/j.aim.2022.108268.

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. ***Theorie de Topos et Cohomologie Etale des Schemas I, II, III***. Vol. 269, 270, 305. Lecture Notes in Mathematics. SGA4. Springer, 1971.

[BD98] John C. Baez and James Dolan. "Higher-Dimensional Algebra III.n-Categories and the Algebra of Opetopes". In: ***Advances in Mathematics*** 135.2 (1998), pp. 145–206. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1997.1695.

[Bal10] Paul Balmer. "Tensor Triangular Geometry". In: ***Proceedings of the International Congress of Mathematicians 2010 (ICM 2010)***. 2010, pp. 85–112. DOI: 10.1142/9789814324359_0041.

[BS22] Guillaume Boisseau and Paweł Sobociński. "String Diagrammatic Electrical Circuit Theory". In: ***Electronic Proceedings in Theoretical Computer Science*** 372 (Nov. 2022). arXiv:2106.07763 [cs], pp. 178–191. ISSN: 2075-2180. DOI: 10.4204/EPTCS.372.13.

[BB86] Francis Borceux and Gilberte van den Bossche. "Quantales and their sheaves". en. In: ***Order*** 3.1 (Mar. 1986), pp. 61–87. ISSN: 1572-9273. DOI: 10.1007/BF00403411.

[BC93] Francis Borceux and Rosanna Cruciani. "Sheaves on a quantale". In: ***Cahiers de Topologie et Géométrie Différentielle Catégoriques*** 34.3 (1993), pp. 209–228.

[BQ96] Francis Borceux and Carmen Quinteiro. "A theory of enriched sheaves". eng. In: ***Cahiers de Topologie et Géométrie Différentielle Catégoriques*** 37.2 (1996), pp. 145–162.

[Bra14] Martin Brandenburg. "Tensor categorical foundations of algebraic geometry". PhD thesis. University of Munster, Oct. 2014. DOI: 10.48550/arxiv.1410.1716.

[Coe10] Bob Coecke, ed. ***New Structures for Physics***. English. 2011th edition. Lecture Notes in Physics. Heidelberg ; New York: Springer, Oct. 2010. ISBN: 9783642128202.

[CK17] Bob Coecke and Aleks Kissinger. ***Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning***. Cambridge: Cambridge University Press, 2017. ISBN: 9781107104228. DOI: 10.1017/9781316219317.

[CHB19] Karin Cvetko-Vah, Jens Hemelaer, and Lieven Le Bruyn. "What is a noncommutative topos?" In: ***Journal of Algebra and Its Applications*** 18.06 (June 2019), p. 1950107. ISSN: 0219-4988, 1793-6829. DOI: 10.1142/S021949881950107X.

[DS21] Fredrik Dahlqvist and Todd Schmid. "How to write a coequation". In: ***Proceedings of the 9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021)***. arXiv:2109.11967 [cs] type: article. Sept. 2021. DOI: 10.48550/arXiv.2109.11967.

[DL23] Ivan Di Liberti and Fosco Loregian. "Accessibility and presentability in 2-categories". In: ***Journal of Pure and Applied Algebra*** 227.1 (Jan. 2023), p. 107155. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2022.107155.

[FST21] Brendan Fong, David Spivak, and Rémy Tuyéras. "Backprop as functor: a compositional perspective on supervised learning". In: ***Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science***. LICS '19. Vancouver, Canada: IEEE Press, 2021.

[Fri20] Tobias Fritz. "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics". In: ***Advances in Mathematics*** 370 (Aug. 2020), p. 107239. ISSN: 0001-8708. DOI: 10.1016/j.aim.2020.107239.

[Gha+18] Neil Ghani et al. "Compositional Game Theory". In: ***Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science***. LICS '18. Oxford, United Kingdom: Association for Computing Machinery, 2018, pp. 472–481. ISBN: 9781450355834. DOI: 10.1145/3209108.3209165.

[Gir87] Jean-Yves Girard. "Linear logic". en. In: ***Theoretical Computer Science*** 50.1 (Jan. 1987), pp. 1–101. ISSN: 0304-3975. DOI: 10.1016/0304-3975(87)90045-4.

[Gra74] J. W. Gray. "Formal category theory: adjointness for 2-categories". In: ***Lecture Notes in Mathematics*** (1974). DOI: 10.1007/bfb0061280.

[HJ95] Claudio Hermida and Bart Jacobs. "Fibrations with indeterminates: contextual and functional completeness for polymorphic lambda calculi". en. In: ***Mathematical Structures in Computer Science*** 5.4 (Dec. 1995), pp. 501–531. ISSN: 1469-8072, 0960-1295. DOI: 10.1017/S0960129500001213.

[HV19] Chris Heunen and Jamie Vicary. ***Categories for Quantum Theory: An Introduction***. Oxford: Oxford University Press, Nov. 2019. DOI: 10.1093/oso/9780198739623.001.0001.

[Hey10] Hans Heymans. "Sheaves on quantales as generalized metric spaces". AAI3435728 ISBN-13: 9781124342894. PhD thesis. Universiteit Antwerpen (Belgium), 2010.

[Hey14] Hans Heymans. "Sheaves on involutive quantales: Grothendieck quantales". en. In: ***Fuzzy Sets and Systems***. Special Issue on Enriched Category Theory and Related Topics (Selected papers from the 33rd Linz Seminar on Fuzzy Set Theory, 2012) 256 (Dec. 2014), pp. 117–148. ISSN: 0165-0114. DOI: 10.1016/j.fss.2013.07.008.

[Jac99] B. Jacobs. ***Categorical Logic and Type Theory***. Studies in Logic and the Foundations of Mathematics 141. Amsterdam: North Holland, 1999.

[Jac93] Bart Jacobs. "Comprehension categories and the semantics of type dependency". In: ***Theoretical Computer Science*** 107.2 (Jan. 1993), pp. 169–207. ISSN: 0304-3975. DOI: 10.1016/0304-3975(93)90169-T.

[Joh02] Peter T. Johnstone. ***Sketches of an Elephant: A Topos Theory Compendium***. Vol. 1-2. Oxford Logic Guides. Oxford University Press, 2002. ISBN: 9780198534259.

[KS05] Masaki Kashiwara and Pierre Schapira. ***Categories and Sheaves***. en. 2005.

[LS88] J. Lambek and P. J. Scott. ***Introduction to Higher-Order Categorical Logic***. en. Cambridge University Press, Mar. 1988. ISBN: 9780521356534.

[Lam12] Joachim Lambek. "Logic and Grammar". In: ***Studia Logica: An International Journal for Symbolic Logic*** 100.4 (2012), pp. 667–681. ISSN: 0039-3215.

[Lob25] Leo Lobski. "Layered Monoidal Theories". PhD thesis. University College London, Dec. 2025. DOI: 10.48550/arXiv.2512.12139.

[Lur08] Jacob Lurie. ***Higher Topos Theory***. Vol. 170. Annals of Mathematics Studies. Princeton University Press, July 2008. DOI: 10.48550/arxiv.math/0608040.

[Mac78] Saunders MacLane. ***Categories for the Working Mathematician***. en. 2nd ed. Graduate Texts in Mathematics. New York: Springer-Verlag, 1978. ISBN: 9780387984032. DOI: 10.1007/978-1-4757-4721-8.

[MM94] Saunders MacLane and Ieke Moerdijk. ***Sheaves in Geometry and Logic: A First Introduction to Topos Theory***. en. Universitext. New York:

Springer-Verlag, 1994. ISBN: 9780387977102. DOI: 10.1007/978-1-4612-0927-0.

[Mah06] Snigdhayan Mahanta. ***On some approaches towards non-commutative algebraic geometry***. Tech. rep. arXiv:math/0501166 type: article. arXiv, Nov. 2006. DOI: 10.48550/arXiv.math/0501166.

[McL95] Colin McLarty. ***Elementary Categories, Elementary Toposes***. Oxford Logic Guides. Oxford, New York: Oxford University Press, Apr. 1995. ISBN: 9780198514732.

[Mel09] Paul-andré Melliès. "Categorical semantics of linear logic". In: ***Interactive Models of Computation and Program Behaviour, Panoramas et Synthèses 27, Société Mathématique de France 1–196***. 2009.

[Res12] Pedro Resende. "Groupoid sheaves as quantale sheaves". en. In: ***Journal of Pure and Applied Algebra*** 216.1 (Jan. 2012), pp. 41–70. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2011.05.002.

[RV22] Emily Riehl and Dominic Verity. ***Elements of ∞-Category Theory***. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2022. ISBN: 9781108837989. DOI: 10.1017/9781108936880.

[Ros25] Ariel E Rosenfield. ***Enriched coverages and sheaves under change of base***. arXiv:2405.19529 [math] version: 4. Apr. 2025. DOI: 10.48550/arXiv.2405.19529.

[SS11] Hisham Sati and Urs Schreiber, eds. ***Mathematical foundations of quantum field theory and perturbative string theory***. Proceedings of symposia in pure mathematics v. 83. Providence, R.I: American Mathematical Society, 2011. ISBN: 9780821851951.

[Ško09] Zoran Škoda. "Some Equivariant Constructions in Noncommutative Algebraic Geometry". In: ***Georgian Mathematical Journal*** 16.1 (2009), pp. 183–202. DOI: doi:10.1515/GMJ.2009.183.

[Str72] Ross Street. "The formal theory of monads". In: ***Journal of Pure and Applied Algebra*** 2.2 (July 1972), pp. 149–168. ISSN: 0022-4049. DOI: 10.1016/0022-4049(72)90019-9.

[Str82] Ross Street. "Two-dimensional sheaf theory". en. In: ***Journal of Pure and Applied Algebra*** 23.3 (Mar. 1982), pp. 251–270. ISSN: 0022-4049. DOI: 10.1016/0022-4049(82)90101-3.

[Str12] Ross Street. ***Monoidal categories in, and linking, geometry and algebra***. arXiv:1201.2991 [math] version: 3 type: article. Oct. 2012. DOI: 10.48550/arXiv.1201.2991.

[SW78] Ross Street and Robert Walters. "Yoneda structures on 2-categories". In: ***Journal of Algebra*** 50.2 (Feb. 1978), pp. 350–379. ISSN: 0021-8693. DOI: 10.1016/0021-8693(78)90160-6.

[TAM25] Ana Luiza Tenório, Caio de Andrade Mendes, and Hugo Luiz Mariano. "On sheaves on semicartesian quantales and their truth values". In: ***Journal of Logic and Computation*** 35.5 (Oct. 2025). arXiv:2204.08351 [math]. DOI: 10.48550/arXiv.2204.08351.

[Ten23] Ana Luiza da Conceição Tenorio. "Sheaves on semicartesian monoidal categories and applications in the quantalic case". en. PhD thesis. Biblioteca Digital de Teses e Dissertações da Universidade de São Paulo, July 2023. DOI: 10.11606/T.45.2023.tde-31082023-163143.

[Vák15] Matthijs Vákár. "Syntax and Semantics of Linear Dependent Types". In: (Jan. 2015). DOI: 10.48550/arxiv.1405.0033.

[Vak17] R Vakil. "The rising sea: foundations of algebraic geometry". 2017.

[Vic14] Steven J. Vickers. "Continuity and geometric logic". In: ***J. Appl. Log.*** 12 (2014), pp. 14–27.

[Wal18] Charles Walker. "Yoneda structures and KZ doctrines". In: ***Journal of Pure and Applied Algebra*** 222.6 (June 2018), pp. 1375–1387. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2017.07.004.